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AN APPROXIMATE ANALYTICAL METHOD OF ANALYSIS OF A THRESHOLD MAINTENANCE POLICY FOR A MULTIPHASE MULTICOMPONENT MODEL

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A multicomponent system is investigated that consists of n identical unreliable components whose nonfailure operating time consists of a number of sequential phases with exponential times. A maintenance policy is studied that proposes the instant replacement of all the components as soon as the number of components that are in some doubtful state (before a failure) amounts to a predefined threshold value. A cost function averaged over a large period is studied. For a fixed n, an analytical approach is considered. If n increases, a new approximate analytical approach is proposed, which is based on results of the type of the averaging principle for recurrent semi-Markovian processes. The conditions of existence and properties of the optimal strategy are studied. An example is considered and possibilities of generalizations are discussed.

Keywords: *multicomponent systems, multistate components, random failures, approximate analytical analysis, threshold maintenance policy, switching processes, recurrent processes of the semi-Markov type.*

1. INTRODUCTION

A multicomponent system is investigated that consists of n identical unreliable components whose nonfailure operating time consists of a number of sequential phases with exponential times. Multicomponent systems with unreliable elements are of special interest for applications in domains related to the control of operation of computer systems, queuing systems, transport networks, aircraft industry, etc. The investigation of multicomponent systems reveals, as a rule, substantial technical problems connected with the dimension of a system, and well-known policies are oriented, as a rule, toward simpler models. The analysis and simulation of such systems becomes considerably complicated if their lifetimes consist of several phases.

In this article, a new approximate analytical approach is proposed to the analysis of threshold maintenance policies of multicomponent systems with a large number n of identical components. The phase state of each of them successively varies according to some Markov process. On time intervals between two sequential maintenances, components function independently of one another. Each failed component is immediately replaced by a new one. The maintenance policy being investigated proposes to replace all the components as soon as the number of components in some "doubtful" state (before a failure) reaches some preassigned threshold value.

Despite the rather intensive investigation of models of preventive maintenance of two-component systems, only a few works are devoted to the analysis of multicomponent systems. We note some most allied works. In an early work [1], the structure of optimal policy was studied for systems with an arbitrary number of components. In [2], a model of a preventive

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maintenance policy of stochastically aging sequential production systems is introduced. In [3], models of preventive maintenance policies for systems with increasing failure rate were investigated. In [4, 5], coordinated group maintenance policies using the number of failed elements in a system were studied. In [6], optimal group maintenance policies are considered for a collection of identical unreliable machines (components), each of which successively passes four probable states. Taking into account that two states are instant, the analysis of this model was reduced to the analysis of the corresponding birth-and-death process. A heuristic approach to the analysis of a model with k states is considered in [7]. Other classes of models and methods of investigation of multicomponent systems are considered in the reviews [8, 9].

In this article, we consider a generalization of the model investigated in [6] to the case of an arbitrary number of states. An analytical approach is proposed, and the asymptotic behavior of a maintenance policy is investigated in the case where the number of components increases. A system is considered that consists of n identical and independently functioning components whose no-failure operation times consist of m sequential exponential states (phases). In this case, it is impossible to analyze the system, using the processes of birth and destruction as in [6]. The state space of the entire system increases as an exponential function of n and m. Therefore, in general, any exact analytical analysis is practically impossible. Nevertheless, in this article, an approximate analytical method is proposed for computation of stationary characteristics of systems with a large n. Some special results are obtained in [10].

The asymptotic method used in this article is based on the results of [11, 12] on the averaging principle for switching processes. The algorithmic approach to the study of some types of block maintenance models (with complete, selective, partial, or cyclic control) for multicomponent semi-Markov systems, including a partially asymptotic analysis of the optimal policy, is considered in [13]. The asymptotic analysis of some block maintenance models of multicomponent semi-Markov systems with small intensities of failures is given in [14].

2. A MODEL AND ANALYSIS FOR A FINITE NUMBER OF COMPONENTS

Let us consider a system consisting of n identical components that operate independently. Each component can be at one of the following $m+1$ probable states: $\{0\}$ is the best one, $\{m\}$ is the failure state, and $\{m-1\}$ is a doubtful state (before a failure). In a state $\{k\}$, $k < m$, a component stays during an exponential time with a parameter λ_k and then passes to the state $\{k+1\}$ with probability $p_{k,\,k+1} = p_k$ or to the state $\{m\}$ (a failure takes place) with probability $1 - p_k$. We assume that $p_{m-1} = 0$. When the component passes to the state $\{m\}$, a corrective maintenance is performed during which the component is immediately set to the state $\{0\}$. This means that we actually see only m states $\{0, 1, \ldots, m-1\}$.

We choose some threshold value $0 < a < 1$ and consider the maintenance policy that is given below and is oriented toward the entire system.

Maintenance Policy. A complete system maintenance (the setting of all the components to the state $\{0\}$) is performed when the number of doubtful components (i.e., the components that are in the state $\{m-1\}$) is greater than or equal to the threshold level na. We denote this policy by $\mathcal{P}(a)$.

We first consider some interval $[0, T]$ on which the complete system maintenance is not performed. Since the state $\{m\}$ is instant, each component on this interval functions irrespective of the other ones in accordance with a Markov process (MP) $x(t)$ with m states $\{0, 1, \ldots, m-1\}$ and the following transition intensities:

$$\lambda_{k,\,k+1} = \lambda_k p_k, \quad \lambda_{k0} = \lambda_k (1 - p_k),$$

$$\lambda_{kj} = 0, \quad j \neq 0, \text{ and } j \neq k+1 \text{ when } 0 \le k \le m-1 \tag{1}$$

and, accordingly, with the intensities of outputs $\lambda_{k,\,k} = \lambda_k$ when $k \neq 0$ and $\lambda_{00} = \lambda_0 p_0$.

If we have

$$0 < \lambda_k < \infty, \quad p_k > 0 \text{ for } k = 0, 1, \ldots, m-2, \tag{2}$$

then the MP $x(t)$ is irreducible and has a stationary distribution $\overline{\pi} = (\pi_0, \ldots, \pi_{m-1})$ that satisfies the following system of algebraic equations:

$$\lambda_k \pi_k = \lambda_{k-1} p_{k-1} \pi_{k-1}, \quad 0 < k \le m-1,$$

$$\lambda_0 p_0 \pi_0 = \sum_{k=1}^{m-1} \lambda_k (1 - p_k) \pi_k, \quad \sum_{i=0}^{m-1} \pi_i = 1. \tag{3}$$

We denote by $\nu_n(i,t)$ the total number of components in a state $\{i\}$ at a moment t. We introduce a vector $\overline{\mu}^{(n)}(t)=(\nu_n(i,t),\ i=0,\dots,m-1),\ t\ge 0$. By construction, $\overline{\mu}^{(n)}(t)$ is a multidimensional MP with a state space

$$Z_n=\{(i_0,i_1,\dots,i_{m-1}),\ i_k=0,1,\dots,\ \sum_{k=0}^{m-1} i_k=n\}.$$

We denote by $\tau_n(a,i_0,\dots,i_{m-1}),\ 0<a<\pi_{m-1}$, the first moment of time at which the number of doubtful components is greater than or equal to na if we first have $\overline{\mu}^{(n)}(0)=(i_0,\dots,i_{m-1})$. This means that

$$\tau_n(a,i_0,\dots,i_{m-1})=\min\{t\colon t>0,\ \nu_n(m-1,t)>na$$

$$\text{for } \overline{\mu}^{(n)}(0)=(i_0,\dots,i_{m-1})\}. \tag{4}$$

For the sake of simplicity, we denote $\tau_n(a)=\tau_n(a,n,0,\dots,0)$. Then, according to the policy $\mathcal{P}(a)$, $\tau_n(a)$ is the length of the regeneration cycle at the end of which the complete system maintenance is performed and all the components turn back to the state $\{0\}$.

The cost function on the interval $[0,\tau_n(a))$ is computed as follows. The cost of a failure of each component (i.e., a component passed to the state 0) equals c_m. At the moment $\tau_n(a)$, a value c_k is paid for each component that is in a state $k,\ k=0,\dots,m-1$, and, for the complete system maintenance, a fixed value C is paid. We denote by $\Sigma_n(a,T)$ the overall cost that is paid on the interval $(0,T]$ according to the policy $\mathcal{P}(a)$. We can also add a cost, for example, D_j, per unit time for each component in a state $j,\ j\le m-1$. However, for simplicity, these costs are not considered.

To investigate the asymptotic behavior of $T^{-1}\Sigma_n(a,T)$ as $T\to\infty$, we need to study the behavior of the cost function on the interval $[0,\tau_n(a)]$, namely, to investigate its expectation and also the expectation of $\tau_n(a)$.

We first investigate some analytical approach for the case where n is fixed. Then the quantity $\tau_n(a,i_0,\dots,i_{m-1})$ can be represented as the time during which the process $\overline{\mu}^{(n)}(t)$ escapes from a domain $D(a)=\{(i_0,i_1,\dots,i_{m-1})\colon(i_0,$ $i_1,\dots,i_{m-1})\in Z_m,\ i_{m-1}\le na\}$, where $Z_m=\{0,1,\dots\}^m$. We denote $G_n(i_0,\ i_1,\dots,i_{m-1})=\mathbf{E}\tau_n(a,i_0,i_1,\dots,i_{m-1})$.

Using the results of [15], it is easily verified that the quantities $G_n(i_0,i_1,\dots,i_{m-1})$ satisfy a system of linear algebraic equations whose solution exists and is unique. The coefficients of this system are computed with direct use of transition intensities of the MP $\overline{\mu}^{(n)}(t)$.

Denote by $H_n(i_0,i_1,\dots,i_{m-1})$ the expectation of the overall cost function on the interval $[0,\tau_n(a)]$, including the cost of system maintenance. Then the quantities $H_n(i_0,i_1,\dots,i_{m-1})$ also satisfy a system of linear algebraic equations whose solution exists and is unique.

Using these results, we can formulate the theorem given below.

THEOREM 1. We assume that relations (2) are true. Then, for the policy $\mathcal{P}(a)$ and for any $n>0$ and $0<a<\pi_{m-1}$, we have

$$\frac{1}{T}\sum\nolimits_n(a,T)\xrightarrow{P}\frac{H_n(n,0,0,\dots,0)}{G_n(n,0,\dots,0)} \tag{5}$$

as $T\to\infty$; the symbol $\xrightarrow{P}$ signifies the convergence in probability.

Proof. Taking into account the policy $\mathcal{P}(a)$, we will construct a renewal reward process as follows. The moments of restoration are the moments of complete system maintenance (the moments of escaping from the domain $D_n(a)$, beginning with the state $(n,0,\dots,0)$). Then the length of the restoration cycle is the quantity $\tau_n(a)$ with the expectation $G_n(n,0,\dots,0)$ and the expectation of the cost (income) during the cycle is $H_n(n,0,\dots,0)$. Then Theorem 1 directly follows from of the law of large numbers for renewal reward processes [16]. $\square$

Theorem 1 gives an algorithm of computation of the limit value of the averaged cost function for any fixed n and any fixed threshold a, and it is this threshold which determines the policy $\mathcal{P}(a)$.

In principle, for each fixed n, we can also numerically compute an optimal threshold a that minimizes the averaged cost function. However, as is easy to see, for a large n, this problem becomes practically unsolvable. Therefore, in the next section, we will propose another approach suitable for analysis of a threshold policy with a large number of components.

3. ASYMPTOTIC RESULTS

We will investigate the case where the number of components increases ($n \to \infty$). We first consider the asymptotic behavior of some additive functionals of a large number of independent Markov systems under transient conditions.

3.1. Analysis of Markov Systems under Transient Conditions. Let $x(t), x_1(t), \dots, x_n(t)$, $t \ge 0$, be identical independent Markov processes with a finite set of states $\{0, 1, \dots, r\}$ and transition intensities $\{\lambda_{ij}, i, j = 0, \dots, r, i \ne j\}$. We introduce the following indicator functions: $\chi_i(j) = 1$ if $i = j$ and $\chi_i(j) = 0$ otherwise. We denote by

$$\nu_n(i, t) = \sum_{k=0}^{n} \chi_i(x_k(t)) \tag{6}$$

the total number of processes in a state $\{i\}$ at a moment t. Let us consider the vector of proportions

$$\bar{\nu}_n(t) = (n^{-1} \nu_n(i, t),\ i = 0, \dots, r). \tag{7}$$

For brevity, we denote a vector $(a_0, \dots, a_r)$ by $\bar{a}$. We introduce a column vector

$$\bar{b}(\bar{q}) = \Big(-q_i \lambda_i + \sum_{k \ne i} q_k \lambda_{ki},\ i = 0, \dots, r \Big),$$

where $\lambda_i = \sum_{j \ne i} \lambda_{ij}$, $\bar{q} = (q_0, \dots, q_r)$, $q_i \ge 0$, $\sum_{i=0}^{r} q_i = 1$. We set $\lambda(\bar{q}) = \sum_{i=0}^{r} q_i \lambda_i$ and assume that the following condition is fulfilled:

$$0 < \lambda_i < \infty,\ i = 0, \dots, r. \tag{8}$$

THEOREM 2. We assume that the MP $x(t)$ is irreducible, condition (8) is fulfilled, and $\bar{\nu}_n(0) \xrightarrow{P} \bar{s}_0$ when $n \to \infty$. Then, for any $T > 0$, we have

$$\sup_{t \le T} |\bar{\nu}_n(t) - \bar{s}(t)| \xrightarrow{P} 0 \tag{9}$$

when $n \to \infty$, where a function $\bar{s}(t) = (s_0(t), \dots, s_r(t))$ satisfies the system of linear differential equations

$$\bar{s}(0) = \bar{s}_0,\quad d\bar{s}(t) = \bar{b}(\bar{s}(t))dt. \tag{10}$$

Proof. We note that $\bar{\nu}_n(t)$ is a multidimensional Markov process with the state space $Q = \{\bar{q}\}$, where $\bar{q} = (q_0, q_1, \dots, q_r) = (j_0 / n,\ j_1 / n, \dots, j_r / n)$, $j_k \ge 0$, $\sum_{k=0}^{r} j_k = n$. Thus, we can use various approaches in investigating the limit behavior of $\bar{\nu}_n(t)$. General theorems of convergence of Markov processes are investigated in [17]. A martingale approach is described in detail in [18]. However, since the character of the process $\bar{\nu}_n(t)$ is recurrent, it makes sense to use in this case the results on the convergence of recurrent semi-Markov processes (RSMPs) (see Appendix), which are obtained in [11, 12] and are oriented toward the analysis of recurrent processes.

In this article, the notations presented in Appendix are used. We note that, as is easy to see, the process $\bar{\nu}_n(t)$ can be described as an RSMP whose duration of stay $\tau_n(\bar{q})$ in a state $\bar{q}$ has an exponential distribution with the parameter $\sum_{k=0}^{r} j_k \lambda_k = n\lambda(\bar{q})$. A quantity $\bar{\xi}_n(\bar{q})$ can be represented in the form

$$\bar{\xi}_n(\bar{q}) = n^{-1}(\bar{e}_k - \bar{e}_i) \text{ with probability } q_i \lambda_{ik} \lambda(\bar{q})^{-1},\quad i, k = 0, \dots, r,\ i \ne k, \tag{11}$$

where $\bar{e}_i$ is the column vector whose ith components is equal to unity and the other components are equal to zero. Thus, we obtain

$$m_n(\bar{q}) = \mathbf{E}\tau_n(\bar{q}) = n^{-1}\lambda(\bar{q})^{-1},\quad \bar{b}_n(\bar{q}) = \mathbf{E}\bar{\xi}_n(\bar{q}) = n^{-1}\bar{b}(\bar{q})\lambda(\bar{q})^{-1}.$$

If we change the time scale and denote $\bar{S}_n(t) = n\bar{\nu}_n(t / n)$, then the process $\bar{S}_n(t)$ stays in the state $\bar{q}$ during some exponentially distributed time $n\tau_n(\bar{q})$ with the parameter $\lambda(\bar{q})$, and the expectation of the value of the jump of $n\bar{\xi}_n(\bar{q})$ equals $\bar{b}(\bar{q})\lambda(\bar{q})^{-1}$.

Next, we note that $\lambda(\bar{q}) \le \max_i \lambda_i$. Let us consider the process $y(t)$ (see Theorem A, relation (42)). Since we have $m(\bar{q}) = \lambda(\bar{q})^{-1} \ge (\max_i \lambda_i)^{-1} > 0$, for any $T < \infty$, we obtain $y(+\infty) > T$. Thus, relation (41) is true for any $T > 0$. But this relation corresponds to convergence (9). Thus, all the conditions of Theorem A are fulfilled, which proves the statement of Theorem 2. $\square$

Note that system (10) is a system of Kolmogorov forward equations for the transition probabilities of the process $x(t)$.

We now investigate the behavior of a cost functional. We consider some generalization and assume that there also exists the possibility of passage from $\{i\}$ to $\{i\}$, i.e., we have $\lambda_{ii} > 0$. Thus, we can consider the case where instantaneous states exist that make it possible to return to the initial state. For such a model, we set $\lambda_i = \sum_{k=0}^{r} \lambda_{ik}$. We note that, in our case, the state $\{m\}$ is instant. Next, we denote by $0 = t_0 \le t_1 \le \dots$ the moments of sequential jumps of $x(t)$. Let $x_k = x(t_k + 0)$ be an embedded Markov chain. We assume that a family of non-negative constants $\{c(i, j),\ i, j = 0, \dots, r\}$ is given. We denote by

$$Z(t) = \sum_{k=0}^{N(t)-1} c(x_k, x_{k+1}) \tag{12}$$

the cost function of the process on $[0, t]$ without taking into account the cost of system maintenance, where $N(t) = \min\{l: l \ge 0, t_{l+1} > 1\}$ is the number of jumps on $[0, t]$. We introduce identical independent Markov processes $x_1(t), x_2(t), \dots, x_n(t),\ t \ge 0$, that are specified in just the same way as $x(t)$. We denote by $Z^{(i)}(t)$ the cost function of a process $x_i(t)$ by analogy with (12). We introduce the averaged cost function of the entire system on $[0, t]$ as follows:

$$Z_n(t) = n^{-1} \sum_{i=1}^{n} Z^{(i)}(t). \tag{13}$$

Now set

$$\tilde{c}(i) = \sum_{j=0}^{r} c(i, j)\lambda_{ij},\ i = 0, \dots, r. \tag{14}$$

THEOREM 3. Let the conditions of Theorem 2 be fulfilled. Then, for any $T > 0$, we have

$$\sup_{t \le T} |Z_n(t) - z(t)| \xrightarrow{P} 0, \tag{15}$$

where

$$z(t) = \int_0^t \sum_{i=0}^{r} \tilde{c}(i) s_i(u) du \tag{16}$$

and the function $\bar{s}(t) = (s_0(t), \dots, s_r(t))$ satisfies the system of equations (10).

Proof. Let us consider the process $(\bar{\nu}_n(t), Z_n(t))$. It also is a multicomponent MP whose set of states is $Q \times R$ and which can be represented as an RSMP (see Appendix); the distribution of the duration $\tau_n(\bar{q})$ of its stay in a state $\bar{q}$ is exponential with the parameter $n\lambda(\bar{q})$ and the value of its jump is represented in the form $(\bar{\xi}_n(\bar{q}), \gamma_n(\bar{q})) = \frac{1}{n}(\bar{e}_j - \bar{e}_i,\ c(i, j))$ with probability $q_i \lambda_{ij} \lambda(\bar{q})^{-1},\ i, j = 0, \dots, r$.

We assume that $g_n(\bar{q}) = \mathbf{E}\, \gamma_n(\bar{q})$. Then we have $g_n(\bar{q}) = n^{-1}\lambda(\bar{q})^{-1}\tilde{g}(\bar{q})$, where $\tilde{g}(\bar{q}) = \sum_{i, j=0}^{r} q_i c(i, j)\lambda_{ij}$. By analogy with the proof of Theorem 2 with the use of Theorem A (see Appendix), we obtain that, for any $T > 0$, we have

$$\sup_{t \le T} \{|\bar{\nu}_n(t) - \bar{s}(t)| + |Z_n(t) - z(t)|\} \xrightarrow{P} 0,$$

where the function $(\bar{s}(t), z(t))$ satisfies the system of differential equations

$$d\bar{s}(t) = \bar{b}(\bar{s}(t))dt,\ \ dz(t) = \tilde{g}(\bar{s}(t))dt,\ \bar{s}(0) = \bar{s}_0,\ z(0) = 0.$$

From this we obtain the representation of $z(t)$ in the form (16), which proves the statement of Theorem 3. □

3.2. Asymptotic Analysis of a Threshold Maintenance Policy. Let us consider the model described in Sec. 2 and carry out its asymptotic analysis when $n \to \infty$, taking into account the above notations. We assume that its system maintenance (the replacement of all the components in the system) is not performed and investigate the asymptotic behavior of the process $\bar{\nu}_n(t)$. Note that many important characteristics such as a cost function or moments of system maintenance can be expressed as integral functionals of the trajectory of the process $\bar{\nu}_n(t)$.

We assume that each component operates irrespective of the others as a homogeneous Markov process with a set of states $\{0, 1, \ldots, m-1\}$ and transition intensities specified in (1). We use the result of Theorem 2. In this case, we have $\bar{b}(\bar{q}) = (b_0(\bar{q}), \ldots, b_{m-1}(\bar{q}))$, where

$$b_k(\bar{q}) = -q_k \lambda_k + q_{k-1} \lambda_{k-1} p_{k-1}, \quad 0 < k \le m-1,$$

$$b_0(\bar{q}) = -q_0 \lambda_0 p_0 + \sum_{j=1}^{m-1} q_j \lambda_j (1 - p_j).$$

Statement 1. Let condition (2) be fulfilled. Then, for any $T > 0$, we have

$$\sup_{t \le T} |\bar{\nu}_n(t) - \bar{s}(t)| \xrightarrow{P} 0, \tag{17}$$

where the function $\bar{s}(t) = (s_0(t), \ldots, s_{m-1}(t))$ satisfies the following system of differential equations:

$$s_0(0) = 1, \; s_i(0) = 0, \; i = 1, \ldots, m-1,$$

$$s'_k(t) = -\lambda_k s_k(t) + \lambda_{k-1} p_{k-1} s_{k-1}(t), \quad 0 < k \le m-1, \tag{18}$$

$$s'_0(t) = -\lambda_0 p_0 s_0(t) + \sum_{j=1}^{m-1} \lambda_j (1 - p_j) s_j(t).$$

The proof of Statement 1 directly follows from Theorem 2.

We now investigate the asymptotic behavior of the model with maintenance. Denote by $\bar{\pi} = (\pi_0, \ldots, \pi_{m-1})$ the stationary solution of system (18) that coincides with the solution of system (3). We investigate $\mathcal{P}(a)$, i.e., the threshold maintenance policy introduced above. Since the number of components grows, we normalize the cost constants being used as follows. On the period $[0, \tau_n(a))$, we pay c_m / n for the replacement of each failed component (for its transition to a state m). At the moment $\tau_n(a)$, we pay c_k / n for each component in a state k, $k = 0, \ldots, m-1$, and also pay C for the system maintenance. As before, we denote by $\Sigma_n(a, T)$ the overall cost that is paid for the functioning of the entire system on the interval $[0, T]$. We first investigate the asymptotic behavior of the moment $\tau_n(a)$ as $n \to \infty$.

LEMMA 1. Let condition (2) be fulfilled. If $0 < a < \pi_{m-1}$, then quantities $\tau_n(a)$, $n \ge 1$, are uniformly integrable, i.e., we have

$$\lim_{L \to \infty} \sup_{n \ge 1} \mathbf{E}\, \tau_n(a) \chi(\tau_n(a) > L) = 0. \tag{19}$$

Proof. We first consider one component, for example, $x_1(t)$. If system maintenances are not performed, then $x_1(t)$ is an irreducible Markov process with continuous time with the set of states $\{0, 1, \ldots, m-1\}$, and with transition intensities $\{\lambda_{ij}, i, j = 0, \ldots, m-1, \; i \ne j\}$. Then, for any initial state $\{i\}$, we have $\mathbf{P}(x_1(t) = m-1 \,|\, x_1(0) = i) \to \pi_{m-1}$ as $t \to \infty$. This means that, for a sufficiently small $\varepsilon < \pi_{m-1} - a$, there exists some $T > 0$ such that, for any $i = 0, \ldots, m-1$, we have

$$\mathbf{P}(x_1(T) = m-1 \,|\, x_1(0) = i) > a + \varepsilon. \tag{20}$$

Consider now the vector $\bar{\nu}_n(t)$. Let us prove that there exists some $\delta > 0$ such that, for a large n and for any initial vector $\bar{\nu}_n(0) = \bar{s}_0$, we have

$$\mathbf{P}(n^{-1} \nu_n^{(m-1)}(T) < a) < 1 - \delta. \tag{21}$$

We will use representation (6). Assume that the initial values $x_k(0) = i_k$ are fixed. For simplicity, we denote $\chi_k = \chi_{m-1}(x_k(T))$. Then, using the Chebyshev inequality, we obtain

$$\mathbf{P}(n^{-1}\nu_n^{(m-1)}(T) < a) = \mathbf{P}(\exp\{-n^{-1}\nu_n^{(m-1)}(T)\} > \exp\{-a\})$$

$$\le e^a \mathbf{E}\exp\left\{-n^{-1}\sum_{k=1}^{n}\chi_k\right\} = e^a \prod_{k=1}^{n} \mathbf{E}\exp\{n^{-1}\chi_k\}. \tag{22}$$

We assume that $q_k = \mathbf{E}\chi_k$. Then, by virtue of (20), we have

$$\mathbf{E}\exp\{-\chi_k / n\} = 1 - q_k(1-\exp\{-1/n\}) \le 1-(a+\varepsilon)(1-\exp\{-1/n\}).$$

The inequality $1 - x \le e^{-x}$ implies that the right side in (22) is less than or equal to $\exp\{a-(a+\varepsilon)n(1-\exp\{-1/n\})\}$. For any fixed ε, a sufficiently small value $\delta_0 > 0$ can be chosen such that we obtain $(a+\varepsilon)(1-\delta_0) - a > \varepsilon/2$. Since $n(1-\exp\{-1/n\}) \to 1$, we can choose a sufficiently large value N such that we have $n(1-\exp\{-1/n\}) > 1-\delta_0$ as $n \ge N$. Then, for $n \ge N$, the right side in (22) is less than or equal to $\exp\{-\varepsilon/2\} < 1$, which proves inequality (21).

We denote $k_t = [t/T]$ for any $t > T$ (the symbol $[\cdot]$ denotes the integer part). Now, by virtue of (21), for any initial vector and for all $n \ge N$, we have

$$\mathbf{P}(\tau_n(a) > t) \le \mathbf{P}(n^{-1}\nu_n^{(m-1)}(iT) < a,\ i = 1,\dots,k_t) = \mathbf{P}(n^{-1}\nu_n^{(m-1)}(T) < a)$$

$$\times\prod_{j=2}^{k_t}\mathbf{P}\left(\frac{1}{n}\nu_n^{(m-1)}(jT) < a \mid n^{-1}\nu_n^{(m-1)}(iT) < a,\ i = 1,\dots,j-1\right) \le (1-\delta)^{k_t}. \tag{23}$$

This inequality means that the distribution of $\tau_n(a)$ has a geometrically majorized "tail." As is obvious, this implies that $\lim\limits_{L\to\infty}\ \sup\limits_{n>N}\ \mathbf{E}\,\tau_n(a)\chi(\tau_n(a) > L) = 0$. Note that, for any fixed $n = 1, 2, \dots, N$, the expectation of $\tau_n(a)$ is finite, which implies the truth of the relation $\lim\limits_{L\to\infty}\mathbf{E}\tau_n(a)\chi(\tau_n(a) > L) = 0$. These relations finally imply (19). $\square$

We now investigate the behavior of the cost function. We recall that $p_{m-1} = 0$. The cost constants $c(i, j)$ are computed as follows: $c(i, 0) = c_m$, $i = 0, 1, \dots, m-1$, and $c(i, j) = 0$ for the other i, j. According to (14), we have $\tilde{c}(i) = c_m\lambda_i(1-p_i)$, $i = 0, 1, \dots, m-1$.

We denote by $\bar{s}(t)$, $t \ge 0$, the solution of (18) and assume that $z(t)$ is specified in (16). We introduce the following deterministic functions:

$$\tau_0(a) = \inf\{t: t > 0,\ s_{m-1}(t) \ge a\}, \tag{24}$$

$$R_0(a) = C + z(\tau_0(a)) + \sum_{i=0}^{m-1} c_i s_i(\tau_0(a)). \tag{25}$$

LEMMA 2. Let conditions (2) be fulfilled, and let the quantity a be not the level of a local extremum for the function $s_{m-1}(t)$. Then, when $n \to \infty$, we have

$$\tau_n(a) \xrightarrow{P} \tau_0(a),\quad \mathbf{E}\tau_n(a)^k \to \tau_0(a)^k,\quad k = 1, 2, \dots \tag{26}$$

Proof. According to formula (4), $\tau_n(a)$ is the time during which the random process $n^{-1}\nu_n(m-1, t)$ reaches the level a. Since $n^{-1}\nu_n(m-1, t)$ uniformly converges in probability to the function $s_{m-1}(t)$ (see (17)) and a is not the level of a local extremum of $s_{m-1}(t)$, we obtain that $\tau_n(a)$ also converges in probability to $\tau_0(a)$. Let us prove this statement. By construction, we have $s_{m-1}(\tau_0(0)) = a$. By virtue of the continuity of $s_{m-1}(t)$, for any sufficiently small ε, we have $s_{m-1}(\tau_0(a)-\varepsilon) < a < s_{m-1}(\tau_0(a)+\varepsilon)$. Then, according to (17), when $n \to \infty$ and $t \le \tau_0(a) - \varepsilon$, we have the relations $n^{-1}\nu_n(m-1, t) < a$ and $n^{-1}\nu_n(m-1, \tau_0(a)+\varepsilon) > a$. As a result, with probability close to unity, we obtain $\tau_n(a) \in (\tau_0(a)-\varepsilon, \tau_0(a)+\varepsilon)$, which implies that $\tau_n(a) \xrightarrow{P} \tau_0(a)$. Next, relation (19) implies the convergence of first moments and, moreover, relations (23) imply the convergence of each moment of any finite order, which finally proves (26). $\square$

LEMMA 3. If conditions (2) are fulfilled, then, for any $n \ge 1$, we have

$$\mathbf{E}Z_n(t) \le C_* \Lambda_* t \ \text{ and } \ \mathbf{E}Z_n(t)^2 \le C_*^2 (\Lambda_*^2 t^2 + \Lambda_* t), \tag{27}$$

where $\Lambda_* = \max\limits_{0 \le i \le m-1} \lambda_i$ and $C_* = \max\limits_{0 \le i, j \le m-1} |c(i, j)|$.

Proof. Let us consider a Markov process $x(t)$ with transition intensities λ_{ij} (see (1)). We denote by $p_{ij} = \lambda_{ij} \lambda_{ii}^{-1}$, $i, j = 0, 1, \dots, m-1$, the transition probabilities for an embedded MP. We assume that the initial state i_0 is fixed. We will construct $x(t)$ on the probability space $[0,1]^\infty$ with the help of a sequence of independent quantities U_l, $l \ge 0$, that are uniformly distributed over $[0,1]$ according to the following algorithm. We denote by x_k, $k \ge 0$, the embedded MP and by η_k the duration of stay in the state x_k. Then, for $x_k = i$, we assume that $\sum\limits_{s=0}^{-1} = 0$ and put $x_{k+1} = j$ if $\sum\limits_{s=0}^{j-1} p_{is} < U_{2k} \le \sum\limits_{s=0}^{j} p_{is}$, $j = 0, 1, \dots, m-1$, $\eta_k = -\lambda_{ii}^{-1} \ln U_{2k+1}$.

Next, on the same probability space, we will construct an auxiliary MP $\tilde{x}(t)$ with the same initial state i_0 and the same embedded process x_k but define the duration of stay in the state x_k as $\tilde{\eta}_k = -\Lambda_*^{-1} \ln U_{2k+1}$.

By construction, the trajectories of embedded MPs for $x(t)$ and $\tilde{x}(t)$ coincide. But since $\tilde{\eta}_k \le \eta_k$ for any k, all the moments of jumps of $\tilde{x}(t)$ are less than or equal to the corresponding moments of jumps of $x(t)$. Now let $\tilde{Z}(t)$ denote the following additive functional (see (12)), which is constructed on the trajectory $\tilde{x}(t)$:

$$\tilde{Z}(t) = \sum_{k=0}^{\tilde{N}(t)-1} c(x_k, x_{k+1}),$$

where $\tilde{N}(t)$ denotes the total number of jumps of $\tilde{x}(t)$ on the interval $[0, t]$. Since $c(i, j) \ge 0$, with probability one, we have

$$Z(t) \le \tilde{Z}(t) \le C_* \tilde{N}(t). \tag{28}$$

Note that since the process $\tilde{x}(t)$ stays in each state during some exponentially distributed time with the parameter Λ_*, the process $\tilde{N}(t)$ is equivalent to a Poisson process with the parameter Λ_*. Then it follows from (28) that

$$\mathbf{E}Z(t) \le C_* \Lambda_* t, \quad \mathbf{E}Z(t)^2 \le C_*^2 (\Lambda_*^2 t^2 + \Lambda_* t). \tag{29}$$

According to (13), it follows from (29) that we have $\mathbf{E}Z_n(t) \le C_* \Lambda_* t$ and $\mathbf{E}Z_n(t)^2 \le C_*^2 (\Lambda_*^2 t^2 + n^{-1} \Lambda_* t)$. From these relations, relations (27) finally follow. □

THEOREM 4. If relations (2) and the condition $0 < a < \pi_{m-1}$ are fulfilled and the quantity a is not the level of a local extremum of the function $s_{m-1}(t)$, then we have

$$\lim_{n \to \infty} \frac{H_n(n, 0, 0, \dots, 0)}{G_n(n, 0, 0, \dots, 0)} = \frac{R_0(a)}{\tau_0(a)}. \tag{30}$$

Proof. According to the law of large numbers for renewal reward processes [16] and Theorem 1, it suffices to find the limit of the expectation of the cost function during the cycle as $n \to \infty$. Denote by $R_n(a)$ the cost (reward) during the time. Then we have

$$R_n(a) = C + Z_n(\tau_n(a)) + \sum_{i=0}^{m-1} n^{-1} \nu_n(i, \tau_n(a)) c_i, \tag{31}$$

where the stochastic function $Z_n(t)$ has been introduced in (13). Taking into account relations (9), (15), (25), and (26) and the theorem of convergence of superpositions of stochastic functions [19, p. 145], we obtain

$$R_n(a) \xrightarrow{P} R_0(a). \tag{32}$$

Next, we should prove that $\mathbf{E}R_n(a) \to R_0(a)$. Since functions $n^{-1}\nu_n(i,t)$ are bounded by the value 1, the convergence in probability implies the convergence of expectations. Let us consider the process $Z_n(t)$. According to (27), we obtain

$$\mathbf{E}Z_n(\tau_n(a))^2 \le C_*^2(\Lambda_*^2\mathbf{E}\tau_n^2(a) + \Lambda_*\mathbf{E}\tau_n(a)). \tag{33}$$

As has been proved earlier, the quantities $\mathbf{E}\tau_n^2(a)$ are uniformly bounded with respect to n, whence we have that the quantities $\mathbf{E}Z_n(\tau_n(a))^2$ are also uniformly bounded with respect to n. Then $Z_n(\tau_n(a))$ is uniformly integrable and the weak convergence implies the convergence of expectations. Since the right side in (32) is determined, we have $\mathbf{E}R_n(a) \to R_0(a)$. □

4. ANALYSIS OF THE OPTIMAL THRESHOLD POLICY

Using the results of the previous section, we will propose an approximate analytical approach to the search for the optimal policy. If cost constants are fixed, then this policy depends only on the choice of the threshold a. According to the previous relations, convergence (5) can be written in the form

$$\frac{1}{T}\Sigma_n(a,T) \xrightarrow{P} \frac{\mathbf{E}R_n(a)}{\mathbf{E}\tau_n(a)}, \tag{34}$$

where $R_n(a)$ is specified in (31). We denote by a_n^* and a_0^* the following optimum levels:

$$a_n^* = \arg\inf_\alpha \frac{\mathbf{E}R_n(a)}{\mathbf{E}\tau_n(a)}, \quad a_0^* = \arg\inf_\alpha \frac{R_0(a)}{\tau_0(a)}$$

(in particular, it may be that $a_n^* = 1$).

We will formulate the result that shows that, under some regularity conditions, the level a_n^* is asymptotically equivalent to the level a_0^*. We first investigate the behavior of $\mathbf{E}R_n(a)$ and $\mathbf{E}\tau_n(a)$ in the lemma given below.

LEMMA 4. We assume that, on some interval $[d_1, d_2]$, the function $s_{m-1}(t)$ (see (18)) strictly monotonically increases. We assume that $A_1 = s_{m-1}(d_1)$, that $A_2 = s_{m-1}(d_2)$, and that $s_{m-1}(t) < A_1$ on the interval $[0, d_1)$. Then, on an interval $[\alpha, \beta]$ such that $A_1 < \alpha < \beta < A_2$, the sequences of functions $\mathbf{E}\tau_n(a)$ and $\mathbf{E}R_n(a)$ converges uniformly with respect to a to the functions $\tau_0(a)$ and $R_0(a)$, respectively (see (24) and (25)).

Proof. Let us consider the sequence of functions $\mathbf{E}\tau_n(a)$. Since $s_{m-1}(t)$ varies monotonically, we can use the result of Lemma 2 and obtain that $\mathbf{E}\tau_n(a) \to \tau_0(a)$ for any $a \in [\alpha, \beta]$. By construction, the functions $\mathbf{E}\tau_n(a)$ and $\tau_0(a)$ do not monotonically decrease with respect to a and, hence, their pointwise convergence implies their uniform convergence. Similarly, the sequence of $\tau_n(a)$ converges in probability to $\tau_0(a)$ uniformly with respect to a on any interval $[\alpha, \beta] \in [A_1, A_2]$. Since the function $\mathbf{E}Z_n(\tau_n(a))$ also does not monotonically decrease with respect to a, its pointwise convergence implies its uniform convergence.

Consider now the function $n^{-1}\mathbf{E}\nu_n(i, \tau_n(a))$. We note that all the components are first in the state 0, that the relation $n^{-1}\mathbf{E}\nu_n(i,t) = \mathbf{P}(x(t) = i \mid x(0) = 0) \le 1$ is true, and that this function is uniformly continuous on each finite interval. Since $n^{-1}\nu_n(i,t) \le 1$, for any $c > 0$ and $\varepsilon > 0$, we obtain

$$\limsup_{n\to\infty} \sup_{|a_1 - a_2| \le c} n^{-1}|\mathbf{E}\nu_n(i, \tau_n(a_1)) - \mathbf{E}\nu_n(i, \tau_n(a_2))|$$

$$\le \limsup_{n\to\infty} \sup_{|a_1 - a_2| \le c} n^{-1}\mathbf{E}|\nu_n(i, \tau_n(a_1)) - \nu_n(i, \tau_n(a_2))|$$

$$\times\chi(\sup_{|a_1 - a_2| \le c} |\tau_n(a_1) - \tau_n(a_2)| \le \varepsilon) + \limsup_{n\to\infty} \mathbf{P}(\sup_{|a_1 - a_2| \le c} |\tau_n(a_1) - \tau_n(a_2)| > \varepsilon)$$

$$\le \limsup_{n\to\infty} \sup_{|t_1 - t_2| \le \varepsilon} n^{-1}\mathbf{E}|\nu_n(i, t_1) - \nu_n(i, t_2)|$$

$$+\limsup_{n\to\infty}\mathbf{P}\left(\sup_{|a_1-a_2|\le c}|\tau_n(a_1)-\tau_n(a_2)|>\varepsilon\right) \tag{35}$$

(note that $\sup\limits_{|a_1-a_2|\le c}$ is taken over $a_1, a_2\in[\alpha,\beta]$ and $\sup\limits_{|t_1-t_2|\le\varepsilon}$ is taken over $t_1, t_2\in[d_1, d_2]$).

In accordance with the uniform convergence of $\tau_n(a)$, the second addend on the right side of (35) becomes vanishingly small with $c\to+0$ for any fixed ε. Using formula (6), when $t_1<t_2$ and $t_2-t_1\to 0$, we obtain

$$n^{-1}\mathbf{E}|\nu_n(i,t_1)-\nu_n(i,t_2)|\le n^{-1}\sum_{k=1}^{n}\mathbf{E}|\chi_i(x_k(t_1))-\chi_i(x_k(t_2))|$$

$$=\mathbf{P}(x(t_1)=i,\ x(t_2)\ne i)+\mathbf{P}(x(t_1)\ne i,\ x(t_2)=i)$$

$$\asymp\left(\mathbf{P}(x(t_1)=i)\lambda_{ii}+\sum_{j\ne i}\mathbf{P}(x(t_1)=j)\,\lambda_{ji}\right)(t_2-t_1)+o(t_2-t_1)\to 0\,.$$

It follows from this relation that the first addend on the right side of (35) also becomes vanishingly small with $c\to+0$. This finally implies the uniform convergence of the function $n^{-1}\mathbf{E}\nu_n(i,\tau_n(a))$. These relations and (31) prove the statement of Lemma 4. □

Consider now a sequence of deterministic functions $F_n(t)$, $t\in[a,b]$, and denote by A_n the set of global minimum points on an interval $[a,b]$.

LEMMA 5. We assume that $F_n(t)$ uniformly converges on $[a,b]$ to a continuous function $F_0(t)$ that has a unique global minimum point t_0. Then we have $\sup\limits_{u\in A_n}|u-t_0|\to 0$.

Proof. We assume that there exists a sequence of points $u_n\in A_n$ such that $u_n\not\to t_0$ when $n\to\infty$. Without loss of generality, we can assume that $u_n\to u_0\ne t_0$. But then, taking into account the property of uniform convergence, we obtain that $F_n(u_n)\to F_0(u_0)$. Since $F_n(u_n)\le F_n(t)$, when $n\to\infty$, we obtain $F_0(u_0)\le F_0(t)$ for all $t\in[a,b]$. But this means that u_0 belongs to the set of global minimum points of $F_0(t)$; we have obtained a contradiction with the fact that the point t_0 is unique. The lemma is proved. □

Next, we introduce a function

$$M(t)=C+z(t)+\sum_{i=0}^{m-1}c_i s_i(t), \tag{36}$$

where $z(t)$ is introduced in (16) and $\tilde{c}(i)=c_m\ \lambda_i(1-p_i)$, $i=0,\ldots,m-1$. For any $\varepsilon>0$ such that $2\varepsilon<\pi_{m-1}$, we specify

$$a_n(\varepsilon)=\arg\min_{a\in[\varepsilon,\,\pi_{m-1}-\varepsilon]}\frac{\mathbf{E}R_n(a)}{\mathbf{E}\tau_n(a)} \tag{37}$$

as a minimum point (or a set of minimum points) on the interval $[\varepsilon,\pi_{m-1}-\varepsilon]$.

THEOREM 5. We assume that the function $M(t)/t$ has a unique point t^* that is the point of its global minimum and is not the point of a local maximum of the function $s_{m-1}(t)$, $s_{m-1}(t)<s_{m-1}(t^*)$ when $t<t^*$ and $s_{m-1}(t^*)<\pi_{m-1}$. Then $a^*=s_{m-1}(t^*)$ is the unique global minimum point for $R_0(a)/\tau_0(a)$ and, for any sufficiently small ε such that $\varepsilon<\pi_{m-1}-s_{m-1}(t^*)$, the relation $a_n(\varepsilon)\to a^*$ is true as $n\to\infty$.

Proof. Note that, by construction, on each interval $[\alpha_1,\alpha_2]$ such that the points α_1 and α_2 are not levels of local extremum of the function $s_{m-1}(t)$, the following relation is true:

$$\inf_{t\in[q_1,\,q_2]} M(t)/t\le\inf_{a\in[\alpha_1,\,\alpha_2]} R_0(a)\,/\,\tau_0(a) \tag{38}$$

and, in this case, we have $s_{m-1}(q_i)=\alpha_i$, $i=1,2$. Next, since t^* is not a local maximum point for $s_{m-1}(t)$, we can choose some interval $[d_1,d_2]$ such that $d_1<t^*<d_2$ and on which $s_{m-1}(t)$ strictly monotonically increases. This means that $\tau_0(a)$ is continuous at the point a^*. Since $R_0(a^*)\,/\,\tau_0(a^*)=M(t^*)\,/\,t^*$, it follows from relation (38) that a^* also is

the point of global extremum for $R_0(a)/\tau_0(a)$ on the interval $[\varepsilon, \pi_{m-1}-\varepsilon]$ (without loss of generality, we can assume that ε and $\pi_{m-1}-\varepsilon$ also are not levels of local maximum for $s_{m-1}(t)$). We now choose points A_1 and A_2 that satisfy the conditions of Lemma 4. On the segment $[d_1, d_2]$, the function $s_{m-1}(t)$ has its inverse function and, on the interval $[A_1, A_2]$, the sequence of functions $\mathbf{E}R_n(a)/\mathbf{E}\tau_n(a)$ uniformly converges to $R_0(a)/\tau_0(a)$. Let $a_n(A_1, A_2)$ be the global minimum point of $\mathbf{E}R_n(a)/\mathbf{E}\tau_n(a)$ on the interval $[A_1, A_2]$. Then, according to Lemma 5, we obtain that $a_n(A_1, A_2)$ converges to a^*.

We now show that, for a large n, we have $a_n(\varepsilon) \in (A_1, A_2)$ (see (37)). This means that $a_n(\varepsilon)$ converges to a^*. We assume that there exists a subsequence of points a_n' such that we have

$$\mathbf{E}R_n(a_n')/\mathbf{E}\tau_n(a_n') \le \mathbf{E}R_n(a_n(A_1, A_2))/\mathbf{E}\tau_n(a_n(A_1, A_2)) \tag{39}$$

and $a_n' \to a_0' \notin (A_1, A_2)$. Since $a_0' \le \pi_{m-1}-\varepsilon$, the set of solutions of the equation $s_{m-1}(t) = a_0'$ is bounded (since $s_{m-1}(t) \to \pi_{m-1}$ as $t \to \infty$) and, by virtue of Lemma 2, it is easy to show that all the partial limits of $\tau_n(a_n')$ and $\mathbf{E}\tau_n(a_n')$ belong to this set. But if $\tau_n(a_n') \xrightarrow{P} t'$, then we also have $\mathbf{E}\tau_n(a_n') \to t'$ and $\mathbf{E}R_n(a_n')/\mathbf{E}\tau_n(a_n') \to M(t')/t'$ and, according to our assumptions, we have $t' \ne t^*$. Now let $n \to \infty$ in (39). Then, by virtue of the convergence of $a_n(A_1, A_2)$ to a^*, we obtain that $M(t')/t' \le R_0(a^*)/\tau_0(a^*) = M(t^*)/t^*$. But this contradicts the uniqueness of the point t^* and finally proves Theorem 5. ◻

Thus, Theorem 5 gives a new approximate analytical approach to the search for the optimal threshold policy when n is large. The conditions of the theorem can be checked numerically in each specific case. This reduces the problem of simulation of a system of high dimensionality to a computational investigation of the extremum of the function that is the solution of a system of linear differential equations.

Example. Let us consider the case where $m = 2$ and assume that $\lambda_0 > 0$, $\lambda_1 > 0$, and $0 < p_0 < 1$. Then we can easily solve system (18) and obtain

$$s_0(t) = \lambda^{-1}(\lambda_1 + \lambda_0 p_0 e^{-\lambda t}),\ \ s_1(t) = \lambda^{-1}\lambda_0 p_0(1 - e^{-\lambda t}),$$

where $\lambda = \lambda_0 p_0 + \lambda_1$.

We now assume that cost constants C, c_1, and c_2 are given and that, for simplicity, $c_0 = 0$. Then we have

$$M(t) = C + c_2 \int_0^t (\lambda_0(1-p_0)s_0(u) + \lambda_1 s_1(u))du + c_1 s_1(t).$$

We set $G = \lambda^{-2}\lambda_0 p_0(c_2(\lambda_1 - \lambda_0(1-p_0)) - c_1\lambda)$. It is easy to make sure that $M(t) = C + c_2\lambda^{-1}\lambda_0\lambda_1 t - G(1 - e^{-\lambda t})$. Differentiating $M(t)/t$, we obtain the following equation for the optimum point:

$$1 - e^{-\lambda t}(\lambda t + 1) = CG^{-1}. \tag{40}$$

Let us consider cases given below.

1. We assume that $G > C$. Since $1 - e^{-\lambda t}(\lambda t + 1)$ strictly monotonically increases (its derivative is positive) from zero to unity, the root t^* of Eq. (40) exists and is unique. And since the sign of the derivative of $M(t)/t$ varies from $-$ to $+$ at the point t^*, t^* is the minimum point of $M(t)/t$. Now, from the relation $s_1(t^*) = a^*$, we have the unique optimum level $a^* = \lambda^{-1}\lambda_0 p_0(1 - e^{-\lambda t^*})$ for the sought-for threshold policy.

The conditions of Theorem 5 are fulfilled and we have $a_n^* \to a^*$.

2. If $G \le C$, then there exists no minimum point of the function $M(t)/t$. This means that it would make no sense to use the threshold policy of the type being considered.

Note that, when $\lambda_1 > \lambda_0(1-p_0)$, it is always possible to find a sufficiently large c_2 and sufficiently small C and c_1 such that the condition $G > C$ is fulfilled.

5. GENERALIZATIONS

We first note that if system maintenances are not performed on some interval $[0, T]$, then we have $s_i(t) = \mathbf{P}(x(t) = i \mid x(0) = 0) = \mathbf{E}\chi_i(x_k(t))$ and, hence, system (10) is a system of Kolmogorov forward differential equations. Then, according to the law of large numbers, for a fixed t, we obtain $\bar{\nu}_n(t) \xrightarrow{P} \bar{s}(t)$. This gives another interpretation of the results of Theorem 2. However, such a straightforward idea cannot be immediately applied to the analysis of $\bar{\nu}_n(t)$ and the cost function as processes in time t and also to the analysis of more general cases where components can be interdependent.

However, our approach that uses asymptotic results for RSMPs (see Appendix) can be extended to more general models.

Let us consider a possible generalization when the duration of stay in a state j has an m_j-phase the Erlangian distribution (or even a phase distribution). This leads to the extension of the state space $\{(l, j),\ l = 1, \dots, m_j,\ j = 0, 1, \dots, m-1\}$ of the basic Markov process. Nevertheless, a similar technique (see Theorems 2 and 3) can be used for analysis of additive functionals defined on the extended space.

We can obtain another interesting generalization after discarding the assumption of independence of components and using some principle of distribution of system load. Let us consider, for example, a system in which the intensity of passage from a state i to j depends on the current number of components $\nu_n(i, t)$ in the state i, which can be written as $\lambda_{ij} = \lambda_{ij}(n^{-1}\nu_n(i, t))$. In this case, the components are not independent and basic functionals cannot be represented as sums of independent MPs. Nevertheless, the process $\bar{\nu}_n(t)$ can also be represented as an RSMP and, during analysis, we can use the methods described in Appendix. In this case, the forms of basic analytical relations are similar but functions $\lambda_{ij}(\cdot)$ are used instead of quantities λ_{ij}. The forms of the corresponding relations for cost functionals and optimal policies are also similar.

Thus, using the averaging principle for processes in a semi-Markov environment [12], we can investigate maintenance policies for multicomponent systems under the action of external Markov or semi-Markov environments. The method considered in this article can also be extended to models of partial and selective control that are considered in [13].

6. APPENDIX. THE AVERAGING PRINCIPLE FOR RECURRENT SEMI-MARKOV PROCESSES

Recurrent semi-Markov processes (RSMPs) form a special subclass of so-called switching processes [11, 12, 20]. Here, we give a formulation of the averaging principle for RSMPs.

We first define the class of RSMPs. We assume that, for each $n = 1, 2, \dots$, families of random vector-valued quantities $F_{nk} = \{(\bar{\xi}_{nk}(\bar{\alpha}), \tau_{nk}(\bar{\alpha})), \bar{\alpha} \in R^r\}$, $k \ge 0$, are given that are independent in totality, that assume values in $R^r \times [0, \infty)$, and whose distributions do not depend on the index k. Here, n is the parameter of a series. Let the initial value $\bar{S}_{n0}$ in R^r be given that does not depend on F_{nk}, $k \ge 0$. We assume that

$$t_{n0} = 0,\ t_{nk+1} = t_{nk} + \tau_{nk}(\bar{S}_{nk}),\ \ \bar{S}_{nk+1} = \bar{S}_{nk} + \xi_{nk}(\bar{S}_{nk}),\ k \ge 0, \quad \text{and}$$

$$\bar{S}_n(t) = \bar{S}_{nk} \ \text{ when } \ t_{nk} \le t < t_{nk+1} \ \text{ and } \ t \ge 0\,.$$

A process $\bar{S}_n(t)$ is called an RSMP. This class of processes is introduced in [20] (see also [11,12]).

We note, in particular, that if the distributions of the quantities introduced do not depend on $\bar{\alpha}$, then the moments t_{nk}, $k \ge 0$, form a restorative process and $\bar{S}_n(t)$ is a restorative process with incomes [16]. If, in this case, the distribution of quantities $\tau_{n1}(\bar{\alpha})$ is exponential, then the process $\bar{S}_n(t)$ is a multidimensional Markov process.

We now consider the process on an interval $[0, nT]\,(n \to \infty)$ and assume that its characteristics depend on n in such a manner that the number of moments of switching of t_{nk} converges in probability to infinity. We assume that there are moment functions $m_n(\bar{\alpha}) = \mathbf{E}\tau_{n1}(n\bar{\alpha})$ and $\bar{b}_n(\bar{\alpha}) = \mathbf{E}\bar{\xi}_{n1}(n\bar{\alpha})$. We denote by $|a|$ the modulus of a quantity a or the norm of a vector $\bar{a}$.

THEOREM A. (Averaging principle.) We assume that, for any $N > 0$, we have

$$\lim_{L\to\infty} \limsup_{n\to\infty} \sup_{|\alpha|<N} \{\mathbf{E}\tau_{n1}(n\bar{\alpha})\chi(\tau_{n1}(n\bar{\alpha}) > L) + \mathbf{E}|\bar{\xi}_{n1}(n\bar{\alpha})|\,\chi(|\bar{\xi}_{n1}(n\bar{\alpha})| > L)\} = 0,$$

that if $\max(|\bar{\alpha}_1|, |\bar{\alpha}_2|) < N$, then we have $|m_n(\bar{\alpha}_1) - m_n(\bar{\alpha}_2)| + |\bar{b}_n(\bar{\alpha}_1) - \bar{b}_n(\bar{\alpha}_2)| \le C_N|\bar{\alpha}_1 - \bar{\alpha}_2| + \alpha_n(N)$, where C_N are some bounded constants and $\alpha_n(N) \to 0$ is uniform in the domain $|\alpha_1| \le N$, $|\alpha_2| \le N$, and that there are functions $m(\overline{\alpha}) > 0$ and $\overline{b}(\overline{\alpha})$ such that, as $n \to \infty$, we have $m_n(\overline{\alpha}) \to m(\overline{\alpha})$ and $\overline{b}_n(\overline{\alpha}) \to \overline{b}(\overline{\alpha})$ for any $\overline{\alpha} \in R^r$ and also $n^{-1}\overline{S}_{n0} \xrightarrow{P} \overline{s}_0$. Then we obtain

$$\sup_{0 \le t \le T} |n^{-1}\overline{S}_n(nt) - \overline{s}(t)| \xrightarrow{P} 0, \tag{41}$$

where the function $\overline{s}(t)$ satisfies the differential equation

$$\overline{s}(0) = \overline{s}_0, \quad d\overline{s}(t) = m(\overline{s}(t))^{-1}\,\overline{b}(\overline{s}(t))dt,$$

and T is any positive number such that $y(+\infty) > T$ with probability one, where

$$y(t) = \int_0^t m(\overline{\eta}(u))du, \quad \overline{\eta}(0) = \overline{s}_0, \; d\overline{\eta}(u) = \overline{b}(\overline{\eta}(u))du \tag{42}$$

(we assume that a solution of $\overline{\eta}(u)$ exists on each interval and is unique).

The proof is given in [12, 21].

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